load is applied to the edge beam outer surface. For a clamped edge plate, the P_o load is directly applied to the plate at $r=a$, with no edge beam surrounding the plate. To represent this clamped support case properly, a special type of edge beam is used. The effect described by the $E_B A_B'$ value for the edge beam is a stiffness effect, which provides a direct resistance to the ring load applied to the beam outer edge. Thus, to provide an edge beam that is "equivalent" to the case of a clamped plate, the $E_B A_B'$ value is taken to be zero so that P_o is not resisted by the beam $E_B A_B'$ stiffness. At the same time, a zero slope must be achieved at the plate outer edge. To do this requires the edge beam to possess a very large value of $E_B I_B$ to provide the necessary inertial restraint at the outer edge to keep the radial slope equal to zero. It follows from the definition of α and δ that $\alpha=0$ and $\delta=\infty$ corresponds to an edge beam "equivalent" to the clamped case. It is seen from Fig. 2 that $b_l a=3.83$ for $\delta=\infty$. It follows from Eq. (14) that $P_o=14.68\,D/a^2$ for the clamped case.

The use of an edge beam presents a practical means of increasing the stability of the plate system, leading to savings in weight and materials because: 1) The load P_o is applied directly to the edge beam. The compressive force actually transmitted to the plate is less than P_o. 2) The presence of the edge beam has a stiffening effect on the plate with respect to its rotation capacity at the boundary. These points can best be shown by examples.

Consider a solid circular plate and its surrounding edge beam with the following properties: $a=32$ in. (81.3 cm), $t=1$ in. (2.54 cm), edge beam width $b=1$ in. (2.54 cm), edge beam height $h=4.75$ in. (12.1 cm), $\sigma=1/3$, $E_B=E$. It therefore follows that $\delta=2.98$, $\alpha=0.148$, $b_l a=3.0$ (from Fig. 2), and Eq. (14) gives $P_o=9.88\,D/a^2$. This same plate, if simply supported, buckles at $P_o=4.28\,D/a^2$, and if clamped, has a $P_o=14.68\,D/a^2$ buckling load. If the plate-edge beam system has the values $a=24.5$ in. (62.25 cm), $t=2/3$ in. (1.69 cm), $b=1$ in. (2.54 cm), $h=8.5$ in. (21.60 cm), $E=E_B$ and $\sigma=1/3$, then $\delta=75.2$, $\alpha=0.520$, and $b_l a=3.78$ (from Fig. 2), and Eq. (14) gives $P_o=19.27\,D/a^2$ as the buckling load. For this plate, the force actually transmitted to the plate by the edge beam is $14.29\,D/a^2$. The additional amount of buckling load, $19.27\,D/a^2$ as compared to $14.29\,D/a^2$, is due to the direct stiffness resistance of the edge beam cross section. The force as transmitted to the plate at $r=a$ is still less than the clamped edge case ($14.29\,D/a^2$ as compared to $14.68\,D/a^2$).

The edge beam therefore does provide a practical way to vary the condition of fixity at the outer edge of the plate, and the resulting stability which is achieved in simply attaching an edge beam to the outer edge of the plate can be greater than the stability provided by using the familiar clamped edge support.

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Sustained Small Oscillations in Nonlinear Control Systems

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I. Introduction

CONSIDER the system of first-order equations

$$x' = f(x, \mu) \tag{1}$$

where μ is a parameter, f is assumed analytic in x and μ at the origin of (x, μ) space and $f(0, \mu)=0$. Suppose there exists a family of solutions

$$\bar{x}(t, \mu) \tag{2}$$

such that to each neighborhood of the origin in (x, μ) there corresponds at least one value, μ, such that $[\bar{x}(t, \mu), \mu]$ is contained in that neighborhood. In this ,case, the origin is called a bifurcation point, the family of solutions $\bar{x}(t, \mu)$ is called a bifurcating branch, and the solutions corresponding to fixed values of the parameter μ are called bifurcating solutions.[3]

Example

Consider the scalar equation $x' = x(x-\mu)$. It can easily be seen that $\bar{x}(t, \mu)=\mu$ is a bifurcating branch of solutions corresponding to a bifurcation branch at the origin of the two-dimensional (x, μ) space and that the bifurcating solutions are constant, or steady-state solutions.

In the following, we shall utilize some results from bifurcation theory to investigate the existence of small amplitude periodic behavior in launch vehicle dynamics. It will be assumed that the nonlinearity exists as a cubic term in the rudder response.

Starting with Poincare, there have been a number of important contributors to the theory. Among the early contributors to the theory of periodic bifurcations were Hopf[2] and Friedrichs.[1] We shall follow quite closely the approach given in Sattinger.[4]

II. Bifurcations in System Theory

In addition to the existence of bifurcating solutions, either steady-state or periodic, it is usually necessary in practice to determine their stability properties. The definition of asymptotic stability of steady-state solutions is well known and need not be presented here. However, the definition of orbital stability of periodic solutions is perhaps less well known and is given here for convenience.

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Definition

Let γ denote the closed path $x=p(t)$ in x-space. The periodic solution $p(t)$ is said to be orbitally stable if for each $\epsilon>0$ there exists a corresponding $\delta>0$ such that every solution $x(t)$ of Eq. (33) whose distance from γ is less than δ for $t=0$ is defined and remains at a distance less than ϵ from γ for all $t \leq 0$. It is said to be orbitally asymptotically stable (and γ is said to be a limit cycle) if in addition the distance of $x(t)$ from γ tends to zero as $t\rightarrow\infty$.

Consider now the special case of Eq. (1) given by

$$x' = (A+\mu B)x + N(x) \qquad N(0)=0 \tag{3}$$

where A, B are real $n \times n$ matrices and $N(x)$ is an analytic vector function beginning with nonzero kth-order terms, $k\geq 2$. In addition, assume that the pair (A, B) has one of the following properties:

Property I

The matrix A has a simple eigenvalue at the origin and if $A\phi_0 = A^T\psi_0 = 0$, then $<B\phi_0, \psi_0> \neq 0$, where $<\cdot,\cdot>$ is the dot product of R^n. Further, the matrix $A+\mu B$ has all roots with negative real part when $\mu<0$ and has exactly one unstable root for $\mu>0$.

Property II

The matrix A has a pair of simple, pure imaginary, roots at $\pm i\omega_o$; and if $A\xi_o = i\omega_o\xi_o$, $A^T\eta_o = i\omega\eta_o$ then $<B\xi_o, \eta_o> \neq 0$. Further assume $A+\mu B$ has all its eigenvalues with negative real parts when $\mu<0$ and has exactly one unstable complex pair if $\mu>0$. The following theorem can then be given relative to the preceding properties.

Theorem[1,2,4]

a) Assume that Eq. (3) satisfies Property I and, in addition, assume that $<N(\phi_0), \psi_0> \neq 0$. Then the origin is a bifurcation point and the bifurcating branch consists of asymptotically stable steady-state solutions.

b) Assume that Eq. (3) satisfies Property II and, in addition, that $<N(\xi_0), \eta_0> \neq 0$. Then the origin is a bifurcation point and the bifurcation branch consists of orbitally asymptotically stable periodic solutions.

Remark

The theorem is proven by showing the existence of an analytic one-parameter family of solutions $[x(\epsilon), \mu(\epsilon)]$ satisfying $x(0)=0$, $\mu(0)=0$. The conditions $<N(\phi_0), \psi_0> \neq 0$ and $<N(\xi_0), \eta_0> \neq 0$ then allow the use of the implicit function theorem to solve for ϵ as a function of μ, providing the bifurcating branch.

Remark

Properties I and II both require that the simple eigenvalues, either real or pure imaginary, pass from the left to the right-hand side of the complex plane when $\mu=0$. This, of course, will not occur in general, but a simple transformation will place the system in proper form. Suppose, for example, the crossing over occurs for $\mu=\mu_0$ in Eq. (3). Then set

$$\lambda = \mu - \mu_0, \quad A_1 = A + \mu_0 B, \quad B_1 = B$$

and the new system

$$x' = (A_1 + \lambda B_1)x + N(x)$$

will satisfy the condition.

III. Application to Flexible Body Dynamics

Consider the following system of equations, obtained from the analysis of flexible body dynamics together with rigid body and control system dynamics.

$$\ddot{\phi}_r + c_2\beta = 0$$

$$\ddot{\phi}_b + 2k\zeta\omega\dot{\phi}_b + \omega^2\phi_b = \omega^2 k\beta$$

$$\beta = f(\sigma) = \sigma - a_2\sigma^3$$

$$\sigma = a_0(\phi_r + \phi_b) + a_1(\dot{\phi}_r + \dot{\phi}_b)$$

where

ϕ_r	= attitude angle of the rigid body
β	= rudder (control force) deflection
c_2	= control effectiveness coefficient
a_0, a_1	= control system gains
ϕ_b	= attitude angle due to bending dynamics (fixed mode)
ζ, ω	= damping and natural frequency of bending mode
k	= normalization constant
σ	= control command
$f(\sigma)$	= nonlinear control command

Letting

$$x = \begin{bmatrix} \phi_r \\ \dot{\phi}_r \\ \phi_b \\ \dot{\phi}_b \end{bmatrix}$$

Eq. (4) can be put into the form

where

$$x' = (A+\mu B)x + b\sigma^3$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -c_2a_0 & -c_2a_1 & -c_2a_0 & -c_2a_1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega^2a_0 & \omega^2a_1 & \omega^2a_0 & \omega^2a_1 - 2\zeta\omega \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ c_2a_2 \\ 0 \\ -ka_2\omega^2 \end{bmatrix}$$

imaginary pair of complex eigenvalues at $\pm j\omega$, providing $c_2 a_1 > 0$. It remains to enforce the conditions of property II. To find ξ_0 and η_0 it is necessary to solve

$$A\xi_0 = i\omega\xi_0 \quad \text{and} \quad A'\eta_0 = i\omega\eta_0$$

Carrying out the calculation, it turns out that

$$\xi_0 = \begin{bmatrix} b_1 \\ b_2 \\ 1 \\ i\omega \end{bmatrix} \qquad \eta_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ i\omega \end{bmatrix}$$

where b_1, b_2 are complex constants depending on a_0, a_1, ω, and c_2. Consequently, the dot product conditions become

$$< B\xi_0, \eta_0 > = 2\zeta\omega \neq 0$$

and $< N(\xi_0), \eta_0 > \neq 0$ if

$$3a_1^2\omega^2 \neq (a_0 b_1 + a_0 + a_1 b_2)^2$$

$$a_0 b_1 + a_0 + a_1 b_2 \neq 0$$

Conclusion

In the preceding, we have assumed a cubic nonlinearity in the rudder dynamics and have determined conditions under which a bifurcating branch of orbitally stable periodic solutions will exist. In the case considered, it was possible to determine rather easily conditions under which the system matrix had a pair of simple, pure imaginary, eigenvalues. In more complicated cases, this can still be accomplished by utilizing various linear stability techniques. The D-decomposition method of determining stability regions[5] ought to prove especially useful in this application.

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Technical Comments

Comment on "Inclusion of Transverse Shear Deformation in Finite Element Displacement Formulations"

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IN a recent Note,[1] the stiffness matrix of a beam element including the shear deformation has been derived by a displacement formulation. The interesting point in that derivation is the necessity to distinguish between the first derivative of the transverse displacement and the rotation of the normal to the cross section of the beam. It is interesting to show that the same result can be easily obtained by using the flexibility matrix of a beam finite element.

The relations between coordinate displacements and forces in Fig. 1 are given by

$$\delta = \alpha S \tag{1}$$

where the flexibility matrix α is obtained from the following expression for the stress energy

$$U' = \frac{1}{2EI}\int_0^a M^2 dx + \frac{1}{2kGA}\int_0^a V^2 dx \tag{2}$$

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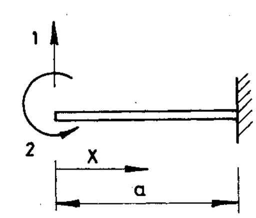

Fig. 1 Coordinates S, δ, for a beam finite element.

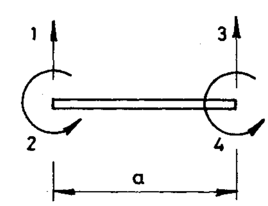

Fig. 2 Coordinates F, u, for a free beam finite element.

From Eq. (2) one obtains

$$[\alpha] = \frac{a}{EI}\begin{bmatrix} a^2/3 + g & -a/2 \\ -a/2 & 1 \end{bmatrix} \tag{3}$$

where, as in Ref. 1, $g = EI/kGA$ and k is the shear factor. From equilibrium considerations, the relationship between the forces in Figs. 1 and 2 is given by

$$F = \beta S \tag{4}$$

where

$$[\beta] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ a & -1 \end{bmatrix} \tag{5}$$